Protection of Life Insurance Companies in a Market-Based Framework

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In this article, we examine to what extent Life Insurance Policyholders can be described as standard Bondholders. Our analysis extends the ideas of Bühlmann [2004], and sequences the fundamental advances of Merton [1974], Longstaff and Schwartz [1995], and Briys and de Varenne [1994, 1997b, 2001]. In particular, we develop a setup where life insurance policies are comparable to hybrid bonds but not to standard risky bonds (as done in most papers dealing with the pricing of participating contracts). In this mixed framework, policyholders are only partly protected against default consequences. Continuous and discrete protections are also studied in an early default Black and Cox [1976] type setting. A comparative analysis of the impact of various protection schemes on ruin probabilities and severities of a Life Insurance company concludes this work.

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Introduction

The recent failures of AIG and Yamato life are here to show, if necessary, that insurance companies, as any firms, can go bankrupt. However, due to the particular nature of the insurance business, its financial risk management is very specific. The last two decades have seen the emergence of an increasing number of papers bridging the conceptual and practical void between financial and actuarial theories. The new regulatory environment, strongly inspired by anglo-american practices, has also called for further development of market-based pricing tools. See in particular Ballotta, Esposito and Haberman [2006] for a detailed account on the enforcement and implications of the new IAS/IFRS/Solvency II norms and Bühlmann [2004] for an insight into market valuation.

The framework adopted here dates back to the analysis of the corporation initiated by Merton [1974]. The essence of this approach is understanding equity as a call-option on the firm's assets, and risky debt as the sum of risk free debt plus a short position on a "default" put on the assets. This approach is also the one chosen by Briys and de Varenne in their papers on life insurance (see their book [2001] for a general treatment) or by Crouhy and Galai [1991] in their analysis of bank deposit regulation. It has been extended, under an assumption of stochastic interest rates, by Bernard, Le Courtois and Quittard-Pinon [2005] in the wider Black and Cox [1976] context that enlarges Merton's framework by considering that bankruptcy is possible at any moment. This study also builds on the framework of Bühlmann [2004] where the relevance of replication arguments is highlighted.

Among the related literature, we can cite Schweizer [2001] who proposes a financial valuation principle that is derived from traditional actuarial premium calculations, but at the same time takes into consideration the possibility of trading in a financial market. In a similar vein, Boyle and Tian [2008] take into account the profit margin and the safety loading in the pricing of Equity Indexed Annuities. These contracts are very similar to the participating policies we are studying. There is a minimum guaranteed rate and a participating coefficient. These authors found that the premium paid by an investor is never equal to the market value of the contract, because of the safety loading and profit margin of the company. They propose to use for the pricing of such contracts a minimum guaranteed rate and a participation coefficient lower than in a fair contract.

In this article, we question the idea that life insurance policyholders are short of a default put on the insurer's assets. In other words, we examine to what extent Life Insurance Policyholders can be described as standard Bondholders, according to the lines of Merton and followers. Indeed, it appears doubtful that participating policies can simply be priced in terms of exotic bonds. In particular, we develop a setup where life insurance policies are comparable to hybrid bonds but not to standard risky bonds (as done in most papers dealing with the pricing of participating contracts). In this mixed framework, policyholders are only partly protected against default consequences.

In the first section, we develop some general insights on the interrelationships between default puts as they are conceived in Finance, and security loadings as they are understood in Insurance Markets. In particular, in the second section, we construct a mixed framework where policyholders can be more or less protected against default consequences. Thus, we consider life insurance contracts as hybrid debt where the importance of the security loading is related to the importance of the debt/equity nature of contracts. The third section is dedicated to its extension to the case when bankruptcy can happen at any time. Continuous and discrete protection schemes are studied in this setting. The fourth section proposes a comparison of three distinct types of protection: the mixed protection introduced in this article, a protection consisting of a simple increase of the assets backing the liabilities, and a protection made of Equity Default Swaps. In particular, the impact of these protections on ruin probabilities and severities is analyzed.

1 Towards A Unified Framework

Firstly we review the basic principles of the so-called Mertonian theory of the firm, as conceived by Merton [1974], Black and Cox [1976] and others. Then, we detail the standard applications of this theory in life insurance, as developed by Boyle and Schwartz [1977], and Briys and de Varenne [1994], and extended by Bacinello [2001], Ballotta [2008], Ballotta, Haberman and Wang [2006], Bernard, Le Courtois and Quittard-Pinon [2005], among others. Finally, we question the direct application of the financial theory to life insurance and propose a new paradigm where safety loadings play a central role.

1.1 The Classical Theory of the Firm in Market Finance

Merton [1974] applied the Black-Scholes-Merton model in the context of a simple corporation issuing risky zero-coupon bonds. This theory being orthodox financial foundation, we shall discuss it only very briefly.

Let a simple company starting at time 0, being endowed with Equity E_0, issuing initially the amount D_0 of zero-coupon bonds maturing at time T with a notional principal K. E_0 and D_0 are invested in the lognormally distributed assets A_0 (there is a unique constant interest rate r). The balance sheet of this company is given in Table 1.

Assets	Liabilities
A_0	E_0
	D_0

Table 1: Initial Capital Structure of the Simple(st) Firm

Following Merton, Equity is a call option on the assets of the firm, with strike price the principal K of the debt, and maturity the maturity T of the debt. In respect of the debt, the risky zero-coupon bond is the sum of a risk-free zero-coupon bond and a *short* position in a (so-called "default") put on the assets of the firm, with strike price K and maturity T. In other words, investors buy corporate bonds cheaper (than otherwise equivalent Government bonds) as a consequence of their unlimited liability regarding a possible bankruptcy. They sell a default put to the firm, and expect a positive spread with respect to the Government interest rate as fair compensation.

At this point, two comments become necessary. First, the Mertonian approach can be extended to the case of stochastic interest rates very easily, when the firm issues such a simple debt profile. Second, default can occur only at the maturity of the risky zero-coupon bonds. Indeed, in the situation where the assets process dives down between 0 and T, it can be recognized that bankruptcy will most likely be declared before the maturity of the debt. Allowing for and modeling such a situation is the work of Black and Cox [1976].

The main contribution of the latter paper is shown under the following condition: when the assets of the firm hit an *ad hoc* barrier, the firm defaults. This barrier can be of any kind; very often it corresponds to the discounted value of the principal of the debt, see Longstaff and Schwartz [1995] and Collin-Dufresne and Golstein [2001] for applications in Finance. In conclusion, the Black and Cox framework extends the Merton analysis by allowing bankruptcy to occur at any time during the life of the company. Of course, in this framework, the debt becomes a path-dependent exotic optional position on the assets. Now, our question becomes: how are the Merton and Black and Cox framework reflected in the recent literature on life insurance theory?

1.2 Current Use of the Financial Theory in Life Insurance

The first papers using financial theory in a life insurance context are those of Boyle and Schwartz [1977] and Brennan and Schwartz [1976]. These authors value simple guarantees as options, under a flat interest rates. Since then, various papers have appeared, including the important contributions of Briys and de Varenne [1994] who more fully develop the pricing of participating contracts. The contracts as considered by these authors pay well defined bonuses, as opposed to the bonuses of with-profit contracts. Mortality is also not considered, but this has no conceptual impact on the validity and interest of their papers, provided it is assumed independent of interest rates.

The fundamental idea underlying the above-mentioned literature is that Merton's capital structure of the corporation can be directly translated in insurance. This yields the balance sheet in Table 2 below, where liabilities are composed of the initial capital E_0 and of the initial contribution L_0 by policyholders. E_0 and L_0 together are invested in the assets A_0. The current literature assumes that policyholders (as opposed to stockholders) face full liability with respect to a possible bankruptcy. Thus, in the literature, life insurance policyholders are identical to bondholders, and life insurance contracts can be valued using the standard financial approach (pricing participating contracts then only boils down to pricing particular exotic contracts).

Assets	Liabilities
A_0	E_0
	L_0

Table 2: Initial Capital Structure of a Simple Life Office

We denote by α the proportion of assets initially owned by policyholders ($\alpha = L_0/A_0$). Consider for instance a participating contract guaranteeing at maturity the fixed amount L_T^g and a participation rate δ. In a Mertonian framework (no early bankruptcy allowed), it possess the simple payoff:

$$L_T^g + \delta(\alpha A_T - L_T^g)^+ - (L_T^g - A_T)^+ \qquad (1)$$

From this expression, one readily understands what a participating contract is (within the current paradigm): *a guaranteed amount, plus a long position in a call on the assets, plus a short position in a put on the identical assets.* The call-option corresponds to the participating bonus. The (short) put-option corresponds to a default put, as defined in financial markets. Of course, similar decompositions hold for the Black and Cox [1976] framework.

To recap, the current literature often assumes that policyholders, like any bondholders, are short a default put on the company issuing their respective policies. In contrast, we show throughout the remainder of this paper that policyholders' and bondholders' positions may actually differ. In this respect, safety loadings will be of utmost importance to achieving a better understanding of this problem.

1.3 Adaptation of the Conceptual Paradigm

We start by building a modified optional framework for life insurance, where replication arguments still hold, standard valuation methods are kept, but where the so-called "default puts" are questioned (i.e. where policyholders are no longer considered identical to bondholders), and where the actuarial practice of safety loadings is introduced. To summarize the problem, our concerns may be simply expressed in the following context:

Bondholders know they are betting on the insolvency probability of the firm. They expect additional return to compensate for these risks. Policyholders (especially long-term life insurance) aim at investing in default-free entities. Life companies thereby impose safety loadings on insurance premia to compensate for bankruptcy potential.

An initial simplified approach could be: a life insurance company sells back the default put to its policyholders. The payoff to policyholders is therefore always positive, no bankruptcy is at present possible, in particular because the company charges much more to policyholders at issuance, and due to replication arguments. This additional charge can be interpreted as the safety loading. In the case of the participating contract, as considered in the previous subsection, its payoff could be written as follows:

$$L_T^g + \delta(\alpha A_T - L_T^g)^+ \tag{2}$$

which amounts to a guarantee, plus a simple call option.

This interpretation of the default put already exists in Ballotta [2008], and is further developed in Ballotta, Esposito and Haberman [2006]. However, the latter authors do analyze the impact on the stability of the insurance company of actually charging policyholders the full price of the default put. In their paper, the probability of ruin is still greater than 0 after policyholders pay the additional loading, since no investment strategy is given or no guarantee has been bought. The safety loading is thus invested in the general fund owned by the company, but no protection is directly constructed apart from the increase of the initial fund. The company is thus still subject to potential default risk (see section 4 for a comparison with other approaches).

In contrast, assuming the payoff $(L_T^g - A_T)^+$ can be perfectly duplicated in the market (see the final section of this article for example) and the company initially buys an *ad hoc* put, default risk disappears and the default probability becomes nil. Everything would appear improved under this alternative framework; however, we wish to point out an important problem. Charging the insured an extra charge diminishes their return on investment. For commercial reasons it seems unlikely that the aforementioned safety loading should be fully charged. Risk-return considerations are just as important for people investing in life insurance contracts. Our opinion is that policyholders invest in policies that are more or less protected, depending on their risk and return preferences. On the other hand, Life Offices will guarantee the insured's amount fully, or partly, depending on how much security loading they may levy.

Returning to the problem dealt with in this subsection, our proposed solution is that life insurance companies sell back a portion of the default put to policyholders, but that this part may not be unitary. The higher the portion of the default put sold back, the higher the corresponding security loading. To make this even more explicit, we construct a simple linear model of default puts / security loadings where a protection coefficient ψ

is introduced. PSI stands for Policyholder'S Immunization coefficient. When ψ is equal to zero, the default put is not sold back to policyholders; they remain entirely short of the default put. This is simply the implicit assumption as taken from the existing literature. When ψ is equal to one, the security loading is complete and the whole default put is consequently sold back to policyholders. In this situation, the contract offers a much lower return than under the preceding situation (i.e. the contract is very secure, but very expensive). Our opinion is that the factor ψ has to be strictly bounded between 0 and 1 to model adequately existing insurance practices.

Thus we introduced the parameter ψ that describes the amount of security loading charged by a life insurance company, and we observe that it is proportional to the amount of default put sold back to policyholders. This parameter appears in the payoff as follows:

$$L_T^g + \delta(\alpha A_T - L_T^g)^+ - (1-\psi) \times (L_T^g - A_T)^+ \tag{3}$$

where if $\psi = 0$, one returns to the risky situation, as described by formula (1), and if $\psi = 1$, one arrives at the case of the risk-free but excessively expensive (from the policyholders' viewpoint) situation described in formula (2).

Whatever the value of ψ, we are working with a company whose capital structure can be written down as in Table 3, where S_0 is the market value of the safety loading. On the assets side, one can easily imagine that the new line corresponding to S_0 is a derivative position protecting the managed portfolio (corresponding to A_0). On the liability side, the bankruptcy protection is ultimately assigned to policyholders, since it is of no relevance to shareholders.

Assets	Liabilities
A_0	$E_0 = (1-\alpha)A_0$
S_0	$L_0 = \alpha A_0 + S_0$

Table 3: Initial Capital Structure of a Life Office

The next two sections will examine the impact of this approach [1] on the pricing of

[1] The pricing methodologies used throughout the article are not new. Lots of papers already addressed

standard participating contracts, both in the Merton and Black and Cox contexts. We start with the simplest situation: default being only possible at maturity.

2 Safety Loadings and Default Puts under the Merton Paradigm

In this section, we consider a company issuing simple participating contracts, and we assume that default can occur only on the maturity of these contracts. For the sake of generality we consider stochastic interest rates. Since default can occur only at maturity, we use a Merton-type framework. We first recall the treatment of standard approaches, and then come to the new approach introduced in this article.

2.1 Defaultable Insurance Policies

Consider a participating contract where we keep the notation introduced in the previous section. As before, A_T is the final value of the assets, L_T^g is the guaranteed amount, also taken at time T (the guarantee is assumed to be deterministic with respect to time and thus constant at expiry time T). The parameter α gives the initial leverage of the company, δ is the bonus coefficient, and r is the interest rate stochastic process. We also suppose that a unique premium is initially paid by policyholders to the life insurance company: in this framework, cash flows can occur only at time 0 and at maturity T.

The payoff of this standard participating contract can be expressed as:

$$\Theta_L(T) = \begin{cases} A_T & \text{if } A_T < L_T^g \\ L_T^g & \text{if } L_T^g \leq A_T \leq \frac{L_T^g}{\alpha} \\ L_T^g + \delta(\alpha A_T - L_T^g) & \text{if } A_T > \frac{L_T^g}{\alpha} \end{cases}$$

the pricing issues of participating contracts. Our contribution is a general reflection on these previous works. We propose a way to model the safety loadings that should be charged to policyholders and to explain heterogeneity of prices observable in the real market.

where in the first state bankruptcy is declared and policyholders recover the residual asset value, in the second state only the guaranteed amount is distributed, and in the third - beneficial - situation, a bonus is offered in addition to the guaranteed amount.

The above payoff can be concisely written as:

$$\Theta_L(T) = L_T^g + \delta(\alpha A_T - L_T^g)^+ - (L_T^g - A_T)^+ \tag{4}$$

where one recognizes a long call on the assets (bonus) and a short put on the same assets (default put). This is also the correspondent formula (1).

In a stochastic interest rate environment, the value V_0 of the participating contract can be obtained directly under the risk-neutral measure Q as:

$$V_0 = \mathbb{E}_Q \left[\, e^{-\int_0^T r_s ds} \; \left(L_T^g + \delta \left(\alpha A_T - L_T^g \right)^+ - \left(L_T^g - A_T \right)^+ \, \right) \, \right] \tag{5}$$

To eliminate the stochastic discount factor, we move to the T-forward neutral universe. V_0 can then be expressed as:

$$V_0 = P(0,T) \left[L_T^g + \delta\alpha E_1 - \delta L_T^g E_2 - L_T^g E_3 + E_4 \right] \tag{6}$$

where:

$$E_1 = \mathbb{E}_{\mathrm{Q}_T} \left[A_T \mathbb{1}_{A_T > \frac{L_T^g}{\alpha}} \right]; \qquad E_3 = \mathrm{Q}_T \left[A_T < L_T^g \right];$$

$$E_2 = \mathrm{Q}_T \left[A_T > \tfrac{L_T^g}{\alpha} \right]; \qquad E_4 = \mathbb{E}_{\mathrm{Q}_T} \left[A_T \mathbb{1}_{A_T < L_T^g} \right];$$

where Q_T denotes the forward-neutral probability, and where E_1, E_2, E_3, and E_4 can be computed in closed-form under the hypotheses given above. Indeed, we here assume that the assets follow a geometric Brownian motion and we use a particular Hull and White model for interest rates. The volatility structure is thus exponential. Given $\nu > 0$ and $a > 0$, can be written as:

$$\sigma_P(t,T) = \frac{\nu}{a} \, \left(1 - e^{-a(T-t)} \right) \tag{7}$$

Also note that under the risk-neutral probability measure Q, the assets value, A_t, and the zero-coupon bond price with expiry date T, $P(t,T)$, follow the classical SDEs:

$$\begin{cases} \frac{dA_t}{A_t} & = r_t dt + \sigma dZ^{\mathrm{Q}}(t) \\ \frac{dP(t,T)}{P(t,T)} & = r_t dt - \sigma_P(t,T) dZ_1^{\mathrm{Q}}(t) \end{cases} \tag{8}$$

where $Z^{\mathrm{Q}}(t)$ and $Z_1^{\mathrm{Q}}(t)$ are Q-standard correlated Brownian motions (ρ is their correlation coefficient). Similar expressions also hold in the forward-neutral universe.

We can now conclude in respect of the E_i's. They can be simply expressed as:

$$E_1 = \Phi_1\left(\mathrm{M}_T;\ \sqrt{\mathrm{V}_T};\ \frac{L_T^g}{\alpha}\right); \qquad E_3 = \mathrm{N}\left(\frac{\ln\left(\mathrm{L}_{\mathrm{T}}^{\mathrm{g}}\right) - \mathrm{M}_{\mathrm{T}}}{\sqrt{\mathrm{V}_{\mathrm{T}}}}\right);$$

$$E_2 = \mathrm{N}\left(\frac{\mathrm{M}_{\mathrm{T}} - \ln\left(\frac{\mathrm{L}_{\mathrm{T}}^{\mathrm{g}}}{\alpha}\right)}{\sqrt{\mathrm{V}_{\mathrm{T}}}}\right); \qquad E_4 = \Phi_2\left(\mathrm{M}_T;\ \sqrt{\mathrm{V}_T};\ L_T^g\right);$$

where M_T and V_T are the two moments of the lognormal distribution of $A_T e^{-r_g T}$:

$$\begin{cases} \mathrm{M}_t = & \ln\left(\frac{A_0}{P(0,t)}\right) + \frac{\nu^2}{4a^3} - \left(\frac{\nu^2}{2a^2} + \frac{\rho\sigma\nu}{a} + \frac{\sigma^2}{2} + r_g\right) t - \frac{\nu^2}{4a^3}\, e^{-2at} \\ & \left(\frac{\nu^2}{2a^3} + \frac{\rho\sigma\nu}{a^2}\right) e^{-a(T-t)} - \left(\frac{\nu^2}{a^3} + \frac{\rho\sigma\nu}{a^2}\right) e^{-aT} + \frac{\nu^2}{2a^3} e^{-a(T+t)}, \\ \\ \mathrm{V}_t = & 2\nu\, \frac{\nu + a\rho\sigma}{a^3}\, e^{-at} - \frac{\nu^2}{2a^3}\, e^{-2at} - \frac{3\nu^2}{2a^3} - \frac{2\rho\sigma\nu}{a^2} + \left(\sigma^2 + \frac{2\rho\sigma\nu}{a} + \frac{\nu^2}{a^2}\right) t, \end{cases}$$

where N is the c.d.f. of the centered reduced Gaussian distribution, and where Φ_1 and Φ_2 are defined by:

$$\begin{cases} \Phi_1(m;\sigma;a) & = \mathbb{E}[e^X \mathbb{1}_{e^X > a}] & = \exp\left(m + \frac{\sigma^2}{2}\right) \mathrm{N}\left(\frac{\mathrm{m} + \sigma^2 - \ln(\mathrm{a})}{\sigma}\right) \\ \Phi_2(m;\sigma;a) & = \mathbb{E}[e^X \mathbb{1}_{e^X < a}] & = \exp\left(m + \frac{\sigma^2}{2}\right) \mathrm{N}\left(\frac{\ln(\mathrm{a}) - \mathrm{m} - \sigma^2}{\sigma}\right) \end{cases} \tag{9}$$

with X the Gaussian random variable following the $\mathcal{N}(m, \sigma^2)$ distribution.

It is therefore easy to compute in closed-form the value of a participating contract, even with stochastic interest rates, when default can occur (only) at maturity. Indeed, what can only be questioned is the appropriateness of such an approach.

As claimed in the previous section, the specific participating contract becomes here a type of defaultable bond. However, policyholders may not want to take the position of bondholders, and may require that the life insurance company protects itself from default. Ultimately, the price of this type of protection will of course be borne by the investors who call for it: i.e. policyholders. In the next section, we compute the financial consequences for policyholders and consider a simplified framework where the company is safe and where it utilizes a “full” safety loading.

2.2 Protected Insurance Policies

We now want to determine how to protect policyholders from a bankruptcy occurring specifically at maturity. A financial protection w.r.t. a decrease in the value of the assets is in fact a put option on the same assets A - with a strike of L_T^g and a maturity of T. This put option is sold ultimately to policyholders, and this cancels out the short position they have on the “default put”. In this particular setting, the price of protection is the value of the underlying security loading.

Let us first have a look at how the analytics of our contract change when it is made completely safe. Obviously the contract’s payoff becomes:

$$\widehat{\Theta}_L(T) = \begin{cases} L_T^g & \text{if } A_T < L_T^g \\ L_T^g & \text{if } L_T^g \leq A_T \leq \frac{L_T^g}{\alpha} \\ L_T^g + \delta(\alpha A_T - L_T^g) & \text{if } A_T > \frac{L_T^g}{\alpha} \end{cases} \tag{10}$$

where it can be seen that policyholders are, in all circumstances, truly guaranteed the amount L_T^g. This yields a compact and straightforward expression for the anticipated payoff:

$$\widehat{\Theta}_L(T) = L_T^g + \delta(\alpha A_T - L_T^g)^+$$

Hence, the market value of the secured contract becomes:

$$\widehat{V}_0 = \mathbb{E}_Q\left[\, e^{-\int_0^T r_s ds} \, \left(L_T^g + \delta(\alpha A_T - L_T^g)^+\right)\right]$$

The secured contract's premium $\widehat{V}_0$ is paid at time 0 and is higher than the risky contract's premium V_0 considered in subsection 2.1.

Keeping the assumptions of the previous subsection, one has for $\widehat{V}_0$:

$$\widehat{V}_0 = P(0,T)\left[L_T^g + \delta\alpha E_1 - \delta L_T^g E_2\right] \tag{11}$$

Then, the initial value S_0 of the safety loading (equal to the difference between the value of the secured contract $\widehat{V}_0$ and the risky one V_0) is exactly matched by the initial price of the Merton default put. In particular:

$$S_0 = \mathbb{E}_Q\left[\, e^{-\int_0^T r_s ds}\, (L_T^g - A_T)^+\right]$$

which yields in closed-form:

$$S_0 = P(0,T)\left[L_T^g E_3 - E_4\right]$$

One readily finds: $\widehat{V}_0 = V_0 + S_0$. We assume that V_0, together with the initial investment of equityholders, is used to constitute the assets of the fund ($A_0 = V_0 + E_0$), and that S_0 is used to buy a product yielding the payoff $(L_T^g - A_T)^+$ at time T on the market. If this put on the assets can be found or duplicated in the market, the contract becomes risk-free (its payoff is given by (10)) and the probability of bankruptcy nil. Another possibility is to invest $\widehat{V}_0 = V_0 + S_0$ in the global fund along with the shareholders's initial investment. In the absence of an investment strategy, the default probability is reduced but still positive (see Ballotta, Esposito and Haberman [2006]); thus the contract is still risky.

Let us illustrate the previous discussion with a short numerical example. We specify our model parameters in Table 4.

A_0	σ	T	α	a	ν	$P(0,T)$	ρ	δ	r_g
100	10%	10	0.9	0.4	0.007	0.6703	-0.05	91.68%	2%

Table 4: Model Parameters

First, A_0 stands for the initial assets value of the issuing company. The assets' volatility σ is set at 10%, which corresponds to a standard investment (approximately half in stocks and half in bonds). We assume that the contract maturity T is equal to ten years, and α is the initial participation of the insured in the capital structure of the firm. The parameters a, ν define the zero-coupon volatility, whilst ρ is the correlation coefficient between the asset generating process and the instantaneous interest rate process. Finally r_g is the minimum guaranteed rate and δ is the participating coefficient.

These parameters are set to typical and reasonable values. We chose values for the parameters close to the ones given in Charlier and Kleynen [2005], based on data from the German market. A volatility of 10% corresponds typically to a portfolio comprised of 40% bonds and 60% stocks, according to the former study. Given the different parameters of our framework, the parameters δ and r_g are such that the risky contract sold to policyholders is fair. So we set: $V_0 = \alpha A_0 = L_0 = 90$. Table 5 displays the participating contract values computed using the parameters defined in the Table 4.

V_0	$\widehat{V}_0$	S_0
90	92.42	2.42

Table 5: Results

At first sight, the initial premia of the two contracts V_0 and $\widehat{V}_0$ are close (with $\widehat{V}_0 = V_0 + S_0$). Yet, in relative terms, the two premia are not so close. Indeed, one can observe that $\frac{S_0}{V_0}$ is approximately worth 2.7%. This is substantial considering the impact this can have on the return of the product. A simple approximation would yield an impact of 0.27% in terms of annual return (due to the 10Y maturity of the product), which is compared against a 2% annual guaranteed rate. Indeed, making the company (or a contract) safe is costly, and making it utterly safe is even more so. Of course, a higher σ would entail a higher discrepancy between V_0 and $\widehat{V}_0$. Note that in anticipation of the following subsection, protecting the investment completely has a cost that investors might, or might not, want to bear in full.

To conclude our illustration, we plot in figure 1 the ratio $\frac{S_0}{V_0}$ with respect to the asset volatility σ. Straightforwardly, for a riskier mix of assets, the bankruptcy probability

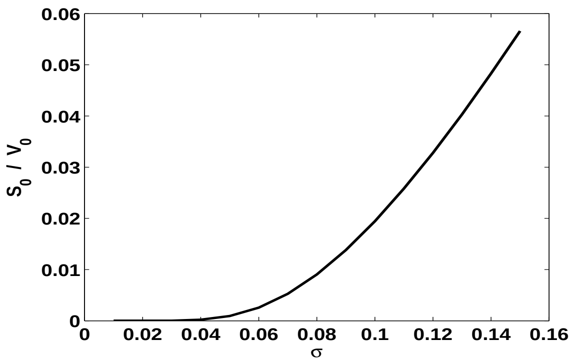

Figure 1: $\frac{S_0}{V_0}$ w.r.t. σ

increases, and so does the safety loading and its ratio to the contract price. Therefore, choosing a higher σ (investing more in stocks and less in bonds for instance) implies levying higher security loadings. This represents of course common intuition.

The contract is here fully protected but the price of perfect coverage is relatively high (having the effect of reducing the appeal of such a contract). In the coming subsection, we consider a mixed situation where opportunity is introduced for smaller safety loadings. We will see that this corresponds to a lower protection of the firm and the insured, and this has an impact on the loss upon default.

2.3 Mixed Approach

We now describe a general linear framework where, upon bankruptcy, policyholders do not recover the entire 'guaranteed amount', but are not completely penalized either by the inferior performance of the assets. The goal of this framework is to model what happens from the policyholders' viewpoint when bankruptcy occurs, depending on the investment and hedging strategy of the insurance company. The goal of this framework is not to give direct investment recommendations ; instead our goal is to provide a better financial understanding of the characteristics of the life insurance firm (section 4 will detail later on the different alternatives an insurance company faces when it invests this safety loading).

We give the following payoffs in the mixed approach:

$$\widehat{\widehat{\Theta}}_L(T) = \begin{cases} \psi\, L_T^g + (1-\psi)\, A_T & \text{if } A_T < L_T^g \\ L_T^g & \text{if } L_T^g \leq A_T \leq \frac{L_T^g}{\alpha} \\ L_T^g + \delta\,(\alpha A_T - L_T^g) & \text{if } A_T > \frac{L_T^g}{\alpha} \end{cases}$$

In the first situation ($A_T < L_T^g$) a mixed amount of the asset value A_T and of the officially guaranteed amount L_T^g is recovered. This state corresponds to the instance where the company could not avoid default, but could, by an appropriate investment strategy, limit the severity of losses, and distribute back more than A_T.

The above payoff can be written in the compact form below:

$$\widehat{\widehat{\Theta}}_L(T) = L_T^g + \delta\,(\alpha A_T - L_T^g)^+ - (1-\psi)\,(L_T^g - A_T)^+ \tag{12}$$

Both payoff expressions are general and return the expressions in subsections 2.1 and 2.2, by assuming respectively $\psi = 0$ and $\psi = 1$. It appears clearly in (12) that the proportion ψ of the default put is sold back to policyholders (meaning that this amount of default put is purchased on the open market by the company to protect itself).

The risk-neutral formula for the contract is obtained straightforwardly as:

$$\widehat{\widehat{V}}_0(\psi) = \mathbb{E}_Q\left[\, e^{-\int_0^T r_s ds}\ \left(L_T^g + \delta\ (\alpha A_T - L_T^g)^+ - (1-\psi)\,(L_T^g - A_T)^+\,\right)\right]$$

where the total default put is still valued according to:

$$S_0 = \mathbb{E}_Q\left[\, e^{-\int_0^T r_s ds}\ (L_T^g - A_T)^+\right]$$

but where the safety loading becomes a fraction ψ of the default put:

$$\widehat{\widehat{S}}_0(\psi) = \mathbb{E}_Q\left[\, e^{-\int_0^T r_s ds}\ \psi\,(L_T^g - A_T)^+\right] = \psi S_0 \tag{13}$$

and where we have the obvious relationship: $\widehat{\widehat{V}}_0 = V_0 + \widehat{\widehat{S}}_0$.

2.4 Degree of Policyholder'S Immunization

This subsection is devoted to different interpretations of the parameter ψ.

First we explain how it is possible to recover the safety loading coefficient ψ of a given company. We omit the different costs related to the marketing of contracts and the management of the company. V_0^m is the price at which a company sells the contract. The market value of a risky contract was previously denoted by $\widehat{\widehat{V}}_0$. The amount $V_0^m - \widehat{\widehat{V}}_0$ is therefore the amount a policyholder spends in addition to the risky contract: it is the safety loading S_0 which in our framework is equal to $\psi \widehat{\widehat{S}}_0$. Thus the simple formula holds:

$$\psi^m = \frac{V_0^m - \widehat{\widehat{V}}_0(\psi = 0)}{S_0}$$

where ψ^m is the target safety loading coefficient.

The parameter ψ can be a comparison tool between different lines of business or different contracts. Indeed, the higher ψ is, the more expensive the contract is. ψ represents the level of safety loading and at the same time the default risk of the insurer. Indeed customers are willing to buy more expensive contracts if these are safer ones.

Second, ψ might be interpreted as a static risk measure directly constrained by regulators. Higher premia mean more protection is sought. Note that

$$\psi S_0 = \psi \mathbb{E}\left[e^{-\int_0^T r_s ds} \left(L_T^g - A_T \right)^+ \right].$$

In case of default (that is $A_T < L_T^g$), the shortfall is $L_T^g - A_T$. Thus ψS_0 is directly linked to the market value of the expected shortfall. This is an important quantity since North American countries recently adopted the CTE (Conditional Tail Expectation) as a criterion. The CTE is the expectation of the loss conditional on the fact that this loss exceeds a limit (typically the Value-at-Risk of the distribution of the loss at some given confidence level α). More details can be found in Hardy [2003], Chapter 9. However, this expectation is done under the historical probability measure if it is used to estimate the solvency capital requirement.

It is interesting to have risk-based rather than flat premia, which is typically the case when premia are independent of the particular features and risks embedded in the product sold (for example when they are only proportional to the actuarial value). Note that premia in our framework include assets' risk (loadings increase if the assets volatility σ increases) and credit risk (in direct proportion to the default put and the shortfall expectation).

Third, rating agencies have clearly an important impact on ψ. Criteria are more and more stringent and ratings also reflect the quality of risk management. Such strategies are part of the evaluations of insurers (see Ingram [2006]). They are based on the way insurers take into account risks in their corporate decision-making. For example, investors will trust wealthy companies and will thus agree to pay them higher premia. Thus, two companies might propose similar products at different though fair prices. The factor ψ reflects the risk exposure of the company in a market where contract prices are competitive. In some sense, it reflects the market value of the protection bought by companies. Thus, similar contracts issued by differently managed companies can be sold at different prices.

Fourth, our modeling of safety loadings also reveals the main difference between financial pricing and insurance pricing. In finance, the no-arbitrage principle holds and prices are uniquely determined and independent of any preferences. In insurance, prices of similar products might differ. Indeed, a risk averse insured prefers to invest in an expensive policy (a policy issued by a more secure vehicle). Can we consider two products identical, when they are identically denominated but sold by differently rated companies? Our answer is no. There is in fact no contradiction between the uniqueness of prices in finance and their apparent multiplicity in insurance. Again, similar products issued by companies protected differently will have different prices. These products although similar can not be considered identical (credit risk is the main difference between them).

We can consider that the parameter ψ also reflects the preference of the issuer since a risk averse insurer will charge more. On the other hand, the risk averse insured are willing to spend enough money when buying their contracts in order to make them safer, because they have no alternative strategies by which to hedge their risks (private investors cannot enter the CDS market, for instance).

The model of this section is not completely realistic, because a company has to be solvent at all times. In particular, the balance sheet of a company is closely monitored by official control authorities. It has to be able to fulfill its commitments towards the insured at any time and not only at maturity. The next section extends the protection of life insurance companies to a continuous-time setting.

3 Safety Loadings and Default Puts under the Black and Cox Paradigm

We extend our discussion to the general situation where default of the insurance company can happen at any time (or more realistically on any given set of audit dates). From a finance viewpoint, this corresponds to building a Black and Cox [1976] type model. We study in this new context the pricing and properties of safety loadings, and, again, interpret them in terms of default options. In fact, we assume a high frequency of regulatory controls and take the continuous limit. To price policies a two-dimension algorithm is necessary to capture the default time and stochastic interest rate effects; we use the one given in Bernard, Le Courtois and Quittard-Pinon [2005] for our analysis.

3.1 Defaultable Insurance Policy

To start with, let us recall how the existing literature prices unprotected participating contracts when default can happen at any time and interest rates are stochastic. Our presentation relies on the expositions in Bernard, Le Courtois and Quittard-Pinon [2005, 2006] where the first paper is dedicated to the pricing of a standard participating contract subject to a constant rate guarantee, and the second paper is devoted to the study of a similar contract subject to a stochastic rate guarantee.

Now let there be, in all situations, a terminal amount $L_T^g = L_0 e^{r_g T}$ guaranteed at maturity T, where r_g is the rate promised to the investors. Note that due to regulatory constraints this rate is often significantly smaller than the rate on treasuries.

As far as the default barrier is concerned, it can be defined as:

$$L_t^g = L_T^g e^{-r_g(T-t)} \tag{14}$$

which is the discounted value at r_g of the terminal guaranteed amount.

On the other hand, the default barrier can also be constructed as follows:

$$L_t^g = L_T^g P(t,T) \tag{15}$$

which is the terminal guaranteed amount discounted against a risk-free zero-coupon bond.

Note that the second instance constructed above imposes a smaller default barrier than the first one. This is because r_g is usually much smaller than a risk-free zero-coupon bond rate; in other words $e^{-r_g(T-t)} >> P(t,T)$. Though $P(t,T)$ is stochastic, in general it will never rise to the level of $e^{-r_g(T-t)}$, due to the small value usually taken by r_g.

Whether one considers a constant or stochastic interest rate guarantee, the default time, in our continuous setting, is always defined as the first time the assets A cross L^g (the default barrier described by one of the above expressions (14) or (15)), so:

$$\tau = \inf\left\{s \in [0,T], A_s < L_s^g\right\}$$

One immediately obtains the generic no-arbitrage price of a participating contract under the risk-neutral probability:

$$\tilde{V}_0 = \mathbb{E}_Q\left[e^{-\int_0^T r_s ds}\left(L_T^g + \delta\left(\alpha A_T - L_T^g\right)^+\right)\mathbb{1}_{\tau>T} + \; e^{-\int_0^\tau r_s ds}\; L_\tau^g\,\mathbb{1}_{\tau\leqslant T}\right] \tag{16}$$

Clearly, if $\tau > T$, default did not happen, and the payoff $L_T^g + \delta\left(\alpha A_T - L_T^g\right)^+$, corresponding to the minimum guarantee plus the participation bonus, is paid at the maturity of the contract. The situation $\tau \leq T$ describes either $\tau = T$, default at maturity, or

$\tau < T$, early default. Restricting oneself to default at maturity reduces to a Merton model, and then correspondingly formula (16) simplifies into (5), and we are once again back to section 2.

What we want to study is the impact and modeling of the condition $\tau < T$. In this state, we suppose that the rebate L^g_τ is paid upon bankruptcy, at the random stopping time τ. This justifies the introduction of the second term in formula (16).

When the guaranteed rate is constant, as with (14), and under a Vasicek specification of r, one can price (16) semi-explicitly as shown by Bernard, Le Courtois and Quittard-Pinon [2005]. The same authors [2006] proved that with a stochastic guaranteed rate, as with (15), formula (16) can be priced in closed-form, still within a Vasicek model of interest rates.

The setting detailed here models and prices participating policies as it would do with exotic bonds. The only difference with respect to subsection 2.1 is the introduction of possible early default: the life insurance company can default at any time between issuance and closing of the contracts, so this is necessarily a more realistic feature. Yet, we are faced with the question of actuarial practices and safety loadings. The coming subsection therefore describes how to protect life-insurance companies and policyholders, in a Black-Cox-Vasicek framework.

3.2 Continuously Protected Insurance Policy

In the early default setting, pricing the default put is a complex path-dependent problem. Indeed, two difficulties arise. The first one is technical, and related to the intrinsic valuation of path-dependent exotic options. The second one is financial and in fact multiple: is the company audited continuously or discretely (at the end of each year for instance)? Does the company want to protect itself discretely or continuously between 0 and T? How does it choose to protect itself and in what proportion? We start our analysis by considering the case where default can happen continuously (at any time between 0 and T), and where the company aims at buying a continuous protection.

The value of a fully protected (continuously between 0 and T, and therefore also at T) participating contract, is always worth:

$$\widehat{V}_0 = \mathbb{E}_Q \left[e^{-\int_0^T r_s ds} \left(L_T^g + \delta \left(\alpha A_T - L_T^g \right)^+ \right) \right] \tag{17}$$

which is the risk-neutral expectation of the guaranteed amount plus bonus, discounted at the risk-free rate.

Theoretically, the price of the total continuous protection (denoted hereafter by G) can be evaluated very easily. Indeed, it suffices to compute the difference between the prices of the fully protected and continuously defaultable contracts. The total continuous protection price is therefore the difference of (17) and (16), which yields after one or two lines of simple calculus:

$$G_0 = \widehat{V}_0 - \tilde{V}_0 = \mathbb{E}_Q \left[\left(e^{-\int_0^T r_s ds} \left(L_T^g + \delta \left(\alpha A_T - L_T^g \right)^+ \right) - \; e^{-\int_0^\tau r_s ds} \; L_\tau^g \right) \mathbb{1}_{\tau \leqslant T} \right] \tag{18}$$

When the barrier is stochastic and defined as in (15), formula (18) can be evaluated in closed-form (see the Appendix for more detail). Working under this assumption, we display our results in Table 6.

$\tilde{V}_0$	$\widehat{V}_0$	G_0
91.34	92.42	1.08

Table 6: Results

Since the framework is unchanged, the totally protected contract's price, $\widehat{V}_0$, is still worth 92.42: see Table 5 for a comparison with previous results. It is interesting to note that $\tilde{V}_0 = 91.34$, the value of the contract that is risky between 0 and T, is bigger than the value $V_0 = 90$ of the contract that is risky only at time T. There would seem to be a paradox here: why would an apparently riskier contract (because of a possible default between 0 and T) be worth more than an apparently less risky contract (defaultable only at maturity T)? The answer is simple: the first of these contracts is in fact the less risky one. This is because early default limits the losses incurred by the company and the insured. The average cost upon default in the Black and Cox context (no surprise is possible, since as soon as the assets are too low, the firm is immediately bankrupt) is less than the average cost in the Merton framework (where one can discover, too late, at time T, that the assets are extremely low). In other words the put hardly has the opportunity to be "in the money" at maturity. Therefore, the premium of the protection or default put is smaller in absolute value in the Black and Cox context w.r.t. the premium in the Merton context. This is why the premium $G_0 = 1.08$ is (less than half) smaller than the premium $S_0 = 2.42$.

Another important consequence can be deduced from our analysis. In the Merton context, a contract defaulted at T only pays back A_T, which can be significantly smaller than L_T^g. The safety loading levied from the insured exists to reduce the severity of ruin and to guarantee an effective amount L_T^g to the insured, even in the case of default (or an amount between A_T and L_T^g, where the proportionality coefficient is ψ). Here, in the Black and Cox setting, because the company is immediately in bankruptcy, the insured recover the guaranteed amount at the time of default τ. They suffer more from a wasted opportunity (of continuing up to T and potentially receiving a bonus) than from a real loss. The previous developments on ψ therefore do not hold. However, this parameter will reappear in the coming paragraphs where the case of the discretely monitored company is considered.

3.3 Discretely Protected Insurance Policy

Now, assume that the balance sheet of the company is monitored at the end of every year: default can be declared only discretely on this set of dates. Therefore, the main concern of the managers of the company is to avoid shortfalls of the assets at the end of each year.

An initial idea is to buy as many puts as there are years in the contract's tenor. This is the simplest way for the company to ensure that it will be solvent at every end of year: each time, its assets A must be over the minimum guaranteed amount (that is $A_{t_i} > L_{t_i}^g$). The payoff of the protection just defined (being a simple series of put options) can be represented as follows:

$$e^{-\int_0^{t_1} r_s ds}\left(L^g_{t_1} - A_{t_1}\right)^+ \; + \; e^{-\int_0^{t_2} r_s ds}\left(L^g_{t_2} - A_{t_2}\right)^+ + ... + e^{-\int_0^{t_n} r_s ds}\left(L^g_{t_n} - A_{t_n}\right)^+$$

Consider for the sake of example the i^{th} put. It admits the following characteristics: a maturity t_i, a strike $L^g_{t_i}$, a final Payoff $\left(L^g_{t_i} - A_{t_i}\right)^+$, and its underlying is of course A.

Let us now compute the value of, the representative i^{th} put with a maturity t_i:

$$\begin{aligned}
\mathbb{E}_Q\left[e^{-\int_0^{t_i} r_s ds}\left(L^g_{t_i} - A_{t_i}\right)^+\right] &= \mathbb{E}_Q\left[e^{-\int_0^{t_i} r_s ds} P(t_i,T)\left(L^g_T - \frac{A_{t_i}}{P(t_i,T)}\right)^+\right] \\
&= P(0,T)\;\mathbb{E}_{Q_T}\left[\frac{P(t_i,T)\left(L^g_T - \frac{A_{t_i}}{P(t_i,T)}\right)^+}{P(t_i,T)}\right] \\
&= P(0,T)\;\mathbb{E}_{Q_T}\left[\left(L^g_T - \frac{A_{t_i}}{P(t_i,T)}\right)^+\right]
\end{aligned}$$

recalling that L^g_T is a constant and that $\frac{A_{t_i}}{P(t_i,T)}$ can be cast in the form $\frac{A_0}{P(0,T)} e^{N_u - \frac{<N>u}{2}}$, where N is a martingale under Q_T and $< N >$ its quadratic variation. This put can be evaluated in closed-form very easily (see the Appendix for more details).

Parameters are chosen as in table 4. The company buys as many annual puts as there are years left in the contract life, that is the company protects itself from default at each year end. In this situation, the protection is very expensive and is equal to 6.85. Indeed, this protection is redundant. Consider then that all the puts actually cover the first period (0 to t_1), all the puts except the first one cover the second period (t_1 to t_2), and so on. Therefore another more refined strategy is necessary to protect the life insurance company in the context of discrete monitoring.

In essence, the appropriate protection has to be path-dependent. Indeed, the discounted payoff of such a protection can be defined as:

$$\begin{aligned}
&e^{-\int_0^{t_1} r_s ds}\left(L^g_{t_1} - A_{t_1}\right)^+ \; + \; e^{-\int_0^{t_2} r_s ds}\mathbb{1}_{A_{t_1} > L^g_{t_1}}\left(L^g_{t_2} - A_{t_2}\right)^+ + \\
&\qquad\qquad ... + e^{-\int_0^{t_n} r_s ds}\mathbb{1}_{A_{t_1} > L^g_{t_1} ... A_{t_{n-1}} > L^g_{t_{n-1}}}\left(L^g_{t_n} - A_{t_n}\right)^+
\end{aligned}$$

and the associated price can be computed by means of Monte Carlo simulations.

In the context of discrete monitoring, a surprise can happen at the end of a particular year, meaning that $A_{t_i} \ll L^g_{t_i}$. On average, the surprise will be less flagrant than waiting for the maturity T (the Merton case). Recall also that in a continuous monitoring situation, no surprise can happen (Black and Cox case, and considering diffusive assets, of course). The first conclusion is that the price of the protection under discrete monitoring should be intermediate between the ones under continuous monitoring and terminal (at maturity) monitoring. The second conclusion is that because surprises are possible under discrete monitoring, it should be possible to introduce a set of parameters ψ_i, in full analogy with the developments of section 2.3. At this point, we believe that it is unnecessary to repeat the same scheme. The ideas of the mixed framework can be extended transparently and directly from the Merton case to the discrete monitoring one.

4 Protection in Practice

A question that often arises is: how can the protection be constructed using market instruments? More precisely, can we find options, swaps or other similar products, in order to directly build the default put and protect the company and the insured? This is what is directly addressed in this section. We also study the impact of using market instruments on the ruin probability, and on the severity of the ruin that the company can incur. Finally, we conclude on the level of protection that the insured may desire.

4.1 Construction of the Protection

There are numerous market instruments that could, apparently, be used by an insurance company such as the one studied in this text. However, when taking a closer look at the possibilities markets offer, it turns out that, very often, the vanilla or slightly exotic options that a life insurance company would consider buying are short maturity products - typically with a one-year maturity. This is clearly not the horizon of an insurance company (we do not consider the possibility of rolling over one-year-maturity derivatives positions). On the other hand, swaps and swaptions are long term products, but they do not necessarily possess payoffs directly meeting the needs of insurance companies.

We could conclude here that it is very difficult to find market instruments to reconstruct the default put, but this would be erroneous. Indeed, a class of products emerged a couple of years ago that possess excellent characteristics with respect to the problem at hand. These products are called equity default swaps. They were created by JP Morgan in 2000 and are in fact insurance policies on equity. Their name mimics the one of credit default swaps, an older and extremely popular product which constitutes half the volume of the credit derivatives market.

In this subsection, we describe the structure of an equity default swap (hereafter EDS), which is typically the instrument that could be used to construct the default put of the company studied in this article. We also give the structure of a credit default swap (hereafter CDS), which can enter upon in the protection of analogous life insurance companies. In the next subsection we consider the impact of using EDSs or CDSs on the risk of the company. Here, we also briefly discuss how the assets can be dynamically managed in order to avoid buying protection in the market. Note though that this alternative approach, which in fact boils down to replicating the put, is expensive since, to be efficient, to needs a high frequency of rebalancing.

EDSs were created for similar reasons than CDSs. They provide protection against a severe equity decline, whereas CDSs provide protection against credit events on a corporate bond. Note that an equity fall of $x\%$ is an event well indentified, whereas credit events are sometimes subject to controversy. EDSs share with CDSs the denomination 'swap'. This is because the investor, who can be considered an insured party, pays his fee in installments rather than as a lump sum. Typical payment periods are six months for EDSs and three months for CDSs. Typical maturities are of five years for both products. The other leg of the swap is the payment to the investor of a rebate when the critical event happens: when the stock loses $x\%$ of its initial value - where x is fixed contractually - for the EDS, or when a credit event occurs for the CDS. EDSs are structured so as to insure very severe drawdowns of the underlying stock: a barrier at 70% of the initial stock value is commonplace. If this event happens, a constant rebate, usually 50% of the initial stock value, is paid to the investor, and, of course, installments are ceased. As far as CDSs are concerned, the rebate compensating for the underlying bond's depreciation upon default is proportional to the loss incured, not a constant.

To sum up, an EDS is merely a deeply out of the money digital put option. It provides protection on equity in a way very suitable to the problem identified in this article, and will be developed in more detail in the coming subsection. On the contrary, CDSs provide protection against more specific triggering events: company bankruptcies, and are therefore adapted to protect corporate bonds, when EDSs are designed to protect stocks. For the sake of simplicity, we will concentrate on simple EDSs and CDSs, that is on products providing protection on a unique stock, or on a unique bond. In practice, insurance companies possess various stocks and bonds in their asset portfolios. Securitized products exist which offer protection on groups of stocks or bonds: ECOs and CDOs. CDOs, or collateralized debt obligations, are now popular products. They correspond to the securitization and tranching of many different CDSs. ECOs, or equity collateralized obligations, are their equity counterparts. EDSs possess the very important property of being medium term products. As mentioned beforehand, their maturity is typically five years, when more standard exotic options mature in a year or less. Therefore these products are quite well adapted for hedging in the insurance business. If necessary, they can be rolled over once or twice in order to match dynamically the maturity of the issued participating contracts; yet, again, interestingly they offer a hedging horizon that is quite long.

Another way to seek protection is to consider an active investment strategy. A lot of work has been devoted to guaranteed funds, see for example Gerber and Pafumi [2000], Basak [1995], El Karoui, Jeanblanc and Lacoste [2005]. A typical and popular strategy is the so-called CPPI, standing for Constant Proportion Portfolio Insurance (see Black and Perold [1992]). The idea is to take advantage of a bull market whilst guaranteeing a minimum level at maturity if the market turns bearish. With the CPPI method, the guaranteed portfolio is built on two financial assets: a risky one and a riskless one. The principle is to invest in the risky asset an amount proportional to the difference between the protected fund value and a floor (F), which could be the present value of the guarantee. The surplus is called the cushion (C), the proportional coefficient is called the multiplier (m) and the amount of risky investment (mC) is the exposure. The multiplier (m) and the floor (F) are strategic parameters that characterize this method. For a large class of stochastic processes (modeling the risky asset dynamics), it is known that the portfolio value is never less than the floor if continuous trading is assumed and general conditions are satisfied. Thus the investor is sure to obtain at maturity at least the guarantee, in our context L_T^g. If the asset price dynamics are governed by a geometric Brownian motion, the cushion admits the same distribution, and the CPPI portfolio value can be expressed in closed form. It is no longer linear in the risky asset price, but is expressed as an m-power of its price (the options considered in this article would become power options in this context). It should be noted that a perfect hedge is only possible if trading takes place continuously. To be more realistic, discrete rebalancing must be introduced. In this new setting the floor can be pierced and the investor, here the insured, is not entirely protected. In this case there is no perfect hedging, nevertheless this new risk can be measured, for example by the conditional tail expectation which can give a basis for the negotiation of a reinsurance contract, or pave the way for the assessment of safety loadings.

4.2 Impact of the Protection

Collected premia depend on the market value of the protection being bought by the company. We discussed earlier in the paper the case where the full default put is sold back to policyholders. In that case, if the market is complete and assets are perfectly replicated, the insurer buys the payoff $(L_T^g - A_T)^+$ and thus completely protects himself against default. However in practice policyholders only partly buy the put: the insurer has to choose how to invest this partial amount in the market.

We study in this subsection the impact of the chosen protection on the ruin probability and severity of ruin of the insurance company. We consider for our illustration three different settings, denoted by ‘a’, ‘b’ and ‘c’. Setting ‘a’ is a theoretical setting: it corresponds to the mixed approach studied in this article where the payoff $\psi\, L_T^g + (1-\psi)\, A_T$ is recovered by policyholders in cases of default. In setting 'a', it is assumed that the levied safety loading is invested in an *ad hoc* put option that can be found in the market.

Then, settings 'b' and 'c' offer practical alternatives, in case the default put mentioned above cannot be found in the market. Setting 'b' explores a natural idea that can be found, for example, in Ballotta, Esposito and Haberman [2006]. This idea is as follows: the safety loading is simply invested in the fund at time 0, more stocks are purchased than without the safety loading by the insurance company to constitute the assets. Finally, setting 'c' describes a protection by means of Equity Default Swaps. To simplify this exposition, computations are done under deterministic interest rates, the maturity of contracts is five years in all three settings, and the EDS' premium is paid at inception and not by installments.

First note that all the computations performed here are done in the historical world. Reasonably, we are interested in real-world ruin probabilities and real-world losses. Remark also that the precise price of the protection chosen is charged to policyholders, so that the contract is in any case fair, and there is no cheating of policyholders by selling a contract not protected as advertised. Also, in the following developments, we consider situations where the fair price at time zero of a specified type of protection is charged to the insured, and study the impact of a change of volatility on these situations.

Let us now give the expressions of the formulas of ruin probabilities and expected losses under the three subsettings:

<u>Setting a:</u>

The ruin probability is simply $P(A_T < L_T^g)$ where $\frac{dA_t}{A_t} = \mu dt + \sigma dz_t$, or equivalently $A_T = A_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma z_T}$. Note that the assets' drift μ needs to be specified, which is traditionally not the case in the risk-neutral world. z is a standard Brownian motion under the historical measure. One thus readily has:

$$P(A_T < L_T^g) = N\left(\frac{\ln\left(\frac{L_T^g}{A_0}\right) - (\mu - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)$$

In case of ruin, the loss incurred by the insured, which can be called severity of ruin, is: $L_T^g - (\psi\, L_T^g + (1-\psi)\, A_T) = (1-\psi)(L_T^g - A_T)$. This is the distance from the barrier, at default. For comparison purposes, we define a severity of ruin that is expected and discounted at time 0. So, our severity of ruin indicator will be defined by:

$$E_P\left(e^{-rT}(1-\psi)(L_T^g - A_T)\mathbb{1}_{A_T<L_T^g}\right)$$

A few lines of simple calculus show that this indicator is equal to:

$$e^{-rT}(1-\psi)\left[L_T^g P(A_T < L_T^g) - \Phi_2\left(\ln(A_0) + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma\sqrt{T}, L_T^g\right)\right]$$

where ϕ_2 is defined in (9).

Setting b:

In this setting, the safety loading $\hat{S}_0$ is not invested to buy a perfect default put, as in setting 'a', but invested in the assets at time 0. In other words, the asset process starts at $A_0' = A_0 + \hat{S}_0$ (this is the approach of Ballotta et al. [2006]). A' constitutes the total assets owned by the insurance company. A is only a part of these assets, on which the contracts are based (for example, bonuses are computed on A, not on A').

The probability of ruin becomes:

$$P(A_T' < L_T^g) = N\left(\frac{\ln\left(\frac{L_T^g}{A_0'}\right) - (\mu - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)$$

The loss upon default is worth $L_T^g - A_T'$, yielding a severity of ruin estimate from time 0:

$$E_P\left(e^{-rT}(L_T^g - A_T')\mathbb{1}_{A_T'<L_T^g}\right)$$

which can be developed as:

$$e^{-rT}\left[L_T^g P(A_T' < L_T^g) - \Phi_2\left(\ln(A_0') + (\mu - \frac{\sigma^2}{2})T, \sigma\sqrt{T}, L_T^g\right)\right].$$

Setting c:

In this setting, the safety loading $\hat{S}_0$ is invested in Equity Default Swaps. The underlying, U, of an EDS is supposed to be representative of the assets of the insurance company, and proportional to them, namely: $\forall t\ U_t = \zeta A_t$ where for a typical insurance company $0 < \zeta \ll 1$. We suppose that the insurance company buys ϕ EDSs (the case $\phi = \frac{1}{\zeta}$ is naturally the complete hedge of A).

Let us now give the payoff of the EDS position. As shown above, an EDS typically pays $U_0/2$ and terminates at the first time τ such that $U_\tau = 70\%\ U_0$. If the underlying does not touch the barrier set at 70% of its initial value, the contract terminates with null value at maturity (the maturity of the EDS is set equal to T, maturity of the contracts issued by the company). The no arbitrage price of the EDS position is therefore:

$$\phi\ E_Q\left(\frac{U_0}{2}\, e^{-r\tau}\, \mathbb{1}_{\tau<T}\right)$$

where $\tau = \inf\{t < T \mid U_t = 0.7 U_0\}$. Note in passing that 'Setting c' is an intertemporal setting, when 'a' and 'b' are not.

A simple proportionality argument yields $\tau = \inf\{t < T \mid A_t = 0.7 A_0\}$. As concerns ϕ, it naturally satisfies:

$$\hat{S}_0 \;=\; \phi\ E_Q\left(\zeta\ \frac{A_0}{2}\, e^{-r\tau}\, \mathbb{1}_{\tau<T}\right) \tag{19}$$

In the present situation, EDSs are bought to limit the severity of ruin beyond a certain level. Ruin can occur in two different manners: at time τ if the company's assets suffer from a severe drawdown (in this situation EDSs are activated), or at time T if the assets never touch the barrier but nevertheless end below L_T^g (in this situation EDSs are not activated). This yields the following ruin probability:

$$P(\tau < T) + P(\tau > T,\ A_T < L_T^g)$$

which can be readily expanded as:

$$N\left(\frac{\ln\left(\frac{L_T^g}{A_0}\right)-\left(\mu-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)+e^{\frac{(2\mu-\sigma^2)\ln(0.7)}{\sigma^2}}N\left(\frac{\ln\left(\frac{A_0}{L_T^g}\right)+2\ln(0.7)+\left(\mu-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

The severity of ruin indicator can be constructed as:

$$E_P\left(e^{-r\tau}\ \max\left(\left[L_\tau^g-(0.7A_0+\phi\zeta A_0)\right],0\right)\ \mathbb{1}_{\tau<T}\right)+E_P\left(e^{-rT}\ \left[L_T^g-A_T\right]^+\ \mathbb{1}_{\tau>T}\right) \quad (20)$$

where $\phi\zeta$ is calibrated from the safety loading (see equation (19)) as follows:

$$\phi\zeta=\frac{\hat{S}_0}{E_Q\left(\frac{A_0}{2}\,e^{-r\tau}\,\mathbb{1}_{\tau<T}\right)}$$

For the sake of brevity, we will not develop in full formula (20), but we will remark that this severity of ruin can be obtained, for instance, using the distribution of τ under the historical measure:

$$dP_\tau(t)=\frac{\ln(0.7)}{\sigma\sqrt{2\pi t^3}}\exp\left(-\frac{\left(\ln(0.7)-\left(\mu-\frac{\sigma^2}{2}\right)t\right)^2}{2\sigma^2 t}\right)\mathbb{1}_{t\geq 0}\,dt.$$

Let us illustrate these three settings with a numerical example. For the sake of simplicity, it is assumed that the assets are made of stocks (in the case of a mix of stocks and bonds, the protection in setting 'c' would use both EDSs and CDSs). We consider a contract whose fair price is $V_0 = 90$; a safety loading equal to 1 is charged. The initial total premium is thus 91.

A_0	T	α	μ	r	S_0	r_g
100	5	0.9	6.5%	5%	1	3.5%

Table 7: Model Parameters

Table 7 gives the parameters used in our illustration. ψ is computed based on (13). The value of the put option being 1.21 (based on the computation of the related risk-neutral expression), and the safety loading being set to 1, one readily has $\psi = 0.82$. As far as δ is concerned, it is set in order to make the contract fair, and therefore depends on the level of the volatility. Because the computations are done in the historical world, it is necessary to specify the drift of the assets in the real world; here we chose $\mu = 6.5\%$.

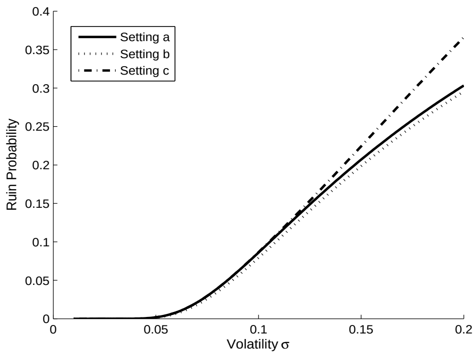

Figure 2: $P(\tau \leq T)$ w.r.t. σ

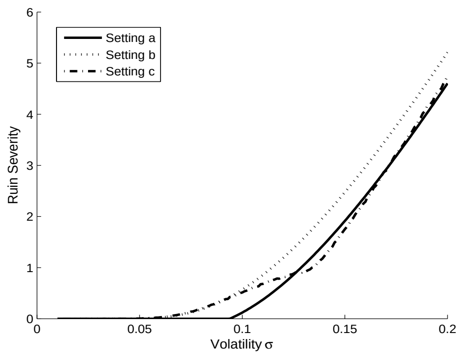

Figure 3: Ruin severity w.r.t. σ

In Figure 2 we represent the ruin probability with respect to the volatility in the three settings, while in Figure 3 we take a look at the severity of ruin, also w.r.t. the company's asset volatility. The amount of safety loading is of course the same in the three settings.

We observe from the graphs that ruin probabilities are comparable in the two settings 'a' and 'b', but that ruin severities are always lower when using default puts instead of investing the safety loading in additional securities.

Let us now take a look at setting 'c'. The ruin probability is higher using EDSs than with other methods. However, one observes the following interesting feature: the ruin severity is smaller with EDSs than with a reinvestment of the safety loading in the assets. Figures 2 and 3 seem to suggest that the pattern of setting 'a', 'b' and 'c' are relatively close for low volatility, but when the volatility is high, the severity of ruin in setting 'b' is higher. This might suggest that for high volatility regime, the investment in EDSs is a good strategy compared to a safety loading fully invested in the assets.

To conclude, the smaller levels of ruin probability and severity are mostly obtained with a protection made of put options. In case these put options are not available or cannot be synthesized in the market, two situations arise. If the ruin probability is the indicator to be minimized, then one should reinvest the safety loading in the assets, as in Ballotta *et al* [2006]. If one is interested in minimizing the severity of ruin, then investing in EDSs will be profitable. Note though that whatever the setting (and if charging only partly the default put), it will not be possible to avoid ruin with certainty.

4.3 How much Protection is Desired

The present article aims at assessing the market value of safety loadings *given a degree of protection ψ chosen by the insurer.* We discussed the impact of choosing such and such protection, in such and such quantity. However, we did not study how ψ, the degree of protection chosen by the insurer, is determined - as this is not one of the goals of the present paper. Let us now however briefly discuss this aspect.

Values of safety loadings are given in the risk-neutral universe, that is, in market value and following the contemporaneous recommendations of regulators such as the IASB. Their prices are arbitrage prices in a complete market, and are therefore completely independent of agent preferences. We insist in the fact that everything cannot be computed in the risk-neutral world whilst neglecting the company's managers preferences. In particular, ψ is not the result of a risk-neutral computation: it is the result of a choice by the managers of the insurance company (perhaps meeting the desires of investors aiming at investing in a more or less risky product). This yields a simple criterion for computing ψ, which can be seen as the solution of the following maximization problem:

$$\max_{\psi}\left(E_P\left[U\left(E_Q\left[e^{-rT}\widehat{\widehat{\Theta}}_L(T)\right]-e^{-rT}\widehat{\widehat{\Theta}}_L(T)\right)\right]\right)$$

where we have the payoff at maturity: $\widehat{\widehat{\Theta}}_L(T)=L_T^g+\delta\left(\alpha A_T-L_T^g\right)^+-(1-\psi)\left(L_T^g-A_T\right)^+$, the total premium at time 0: $E_Q\left[e^{-rT}\widehat{\widehat{\Theta}}_L(T)\right]$, and a choice of utility function U for the manager. This is a simplified way to deal with the choice of ψ. It might also involve costs, the initial wealth of insurer and insured, and the insured's preference V (usually through a constraint on the expected utility of their final wealth). A more detailed estimation of the parameter ψ is left for future research.

Summary and Conclusions

This study is devoted to the calculation of appropriate premia and loadings for participating insurance contracts. We introduce safety loadings in close relationship to default puts on insurance companies. These loadings reflect the asset and credit risks of underlying products. This study also explains why different insurers sell similar contracts at different prices (the difference being a credit risk premium). Loadings may depend on various features, such as the preference of the insurer or the insured, regulation, enterprise risk management, ratings, and credit risk.

The developments of this article shed light on a variety of interesting problems: when a product is reputedly guaranteed, is it indeed wholly guaranteed? What is the market price of a guarantee underlying a life insurance contract? What is the relationship between safety loadings and default puts? How can the financial default framework be applied to insurance? How far is a policyholder differentiated from a bondholder (this is perhaps the most important question, which we answer by means of the mixing parameter ψ)?

We also believe that the approach developed in this article can be applied in other fields, like the one of bank deposit insurance. Indeed, Merton [1977, 1978] showed that bank deposit guarantees are equivalent to default puts on the assets of the bank hosting the deposits. In his first article, there is one final date for monitoring, whilst in the second monitoring can occur at any time and is driven by a Poisson distribution. There are some clear analogies between the guarantees of bank deposits and the guarantees attached to contracts like the ones studied in this article (see also Crouhy and Galai [1991]). Our conclusion - via this example of bank deposit guarantees - is that the developments within this article can be of interest to other subfields of finance and insurance.

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Appendix

Computation of G_0

Recall that:

$$G_0 = \widehat{V}_0 - \tilde{V}_0 = \mathbb{E}_Q\left[\left(e^{-\int_0^T r_s ds}\left(L_T^g + \delta\left(\alpha A_T - L_T^g\right)^+\right) - \ e^{-\int_0^\tau r_s ds}\ L_\tau^g\right)\mathbb{1}_{\tau\leqslant T}\right] \tag{21}$$

Then:

$$\begin{aligned} G_0 = {} & P(0,T)\, L_T^g\, \mathrm{Q}_T\left(\tau \leqslant T\right) \ + \ P(0,T)\,\delta\,\alpha\,\mathbb{E}_{\mathrm{Q}_T}\left[A_T\mathbb{1}_{A_T>\frac{L_T^g}{\alpha},\tau\leqslant T}\right] \\ & - P(0,T)\,\delta\, L_T^g\,\mathbb{E}_{\mathrm{Q}_T}\left[A_T > \frac{L_T^g}{\alpha}, \tau \leqslant T\right] - L_T^g\,\mathrm{Q}_T\left(\tau\leqslant T\right) \end{aligned} \tag{22}$$

The main difficulty here is to show that $\mathbb{E}_Q\left[e^{-\int_0^\tau r_s ds}\, L_\tau^g\, \mathbb{1}_{\tau\leqslant T}\right] = P(0,T)\, L_T^g\, \mathrm{Q}_T\left(\tau \leqslant T\right)$ because the passage from the risk-neutral probability to the forward-neutral one is direct in the first part of formula (21) and simply stems from the definition of these two worlds.

Indeed, one can write:

$$\mathbb{E}_Q\left[e^{-\int_0^\tau r_s ds}\, L_\tau^g\, \mathbb{1}_{\tau\leqslant T}\right] = \ L_T^g\,\mathbb{E}_Q\left[e^{-\int_0^\tau r_s ds}\, P(\tau,T)\,\mathbb{1}_{\tau\leqslant T}\right]$$

where the payoff of $P(\tau,T)\,\mathbb{1}_{\tau\leqslant T}$ is discounted from τ to 0.

Taking as a new numéraire $P(.\,,\,T)$, one can write under Q_T:

$$\mathbb{E}_Q\left[e^{-\int_0^\tau r_s ds}\, L_\tau^g\, \mathbb{1}_{\tau\leqslant T}\right] = \ L_T^g\, P(0,T)\,\mathbb{E}_{\mathrm{Q}_T}\left[\frac{P(\tau,T)\,\mathbb{1}_{\tau\leqslant T}}{P(\tau,T)}\right]$$

which immediately simplifies to:

$$\mathbb{E}_Q\left[e^{-\int_0^\tau r_s ds}\, L_\tau^g\, \mathbb{1}_{\tau\leqslant T}\right] = \ L_T^g\, P(0,T)\,\mathrm{Q}_T\left(\tau\leqslant T\right)$$

and then the result obtains.

Thanks to time change techniques, we obtain closed-form formulas for the three unknown terms in formula (22). To simplify notations, we use the auxiliary functions η^+ and η^- defined by the following expressions:

$$\eta^+(x) = \mathrm{N}\left(\frac{\ln(\mathrm{x}) + \frac{1}{2}\xi(\mathrm{T})}{\sqrt{\xi(\mathrm{T})}}\right) \quad \text{and} \quad \eta^-(\mathrm{x}) = \mathrm{N}\left(\frac{\ln(\mathrm{x}) - \frac{1}{2}\xi(\mathrm{T})}{\sqrt{\xi(\mathrm{T})}}\right)$$

where N denotes the cumulative standard normal distribution function.

$$\begin{cases} \mathrm{Q}_T\left(\tau\leqslant T\right) & = \eta^+(\alpha P(0,T)e^{r_g T}) + \frac{1}{\alpha P(0,T)e^{r_g T}}\ \eta^-(\alpha P(0,T)e^{r_g T}) \\ \mathbb{E}_{\mathrm{Q}_T}\left[A_T\mathbb{1}_{A_T>\frac{l_T^g}{\alpha}}\mathbb{1}_{\tau\leqslant T}\right] & = L_T^g\ \eta^+(\alpha^2 P(0,T)e^{r_g T}) \\ \mathrm{Q}_T\left[A_T > \frac{l_T^g}{\alpha}\ ,\ \tau\leqslant T\right] & = \frac{1}{\alpha P(0,T)e^{r_g T}}\ \eta^-(\alpha^2 P(0,T)e^{r_g T}) \end{cases} \tag{23}$$

$\square$

Computation of the discrete protection

Let us define by $\mathcal{E}$:

$$\mathcal{E} = P(0,T)\mathbb{E}_{\mathrm{Q}_T}\left[\left(L_T^g - \frac{A_{t_i}}{P(t_i,T)}\right)^+\right]$$

Note that:

$$\frac{A_u}{P(u,T)} = \frac{A_0}{P(0,T)}e^{N_u - \frac{1}{2}\xi(u)}, \tag{24}$$

where the differential of N is defined by:

$$dN_s = (\sigma_P(s,T) + \rho\sigma)\, dZ_1^{\mathrm{Q}_T}(s) + \sigma\sqrt{1-\rho^2}\, dZ_2^{\mathrm{Q}_T}(s), \tag{25}$$

and the quadratic variation of N is:

$$\xi(u) \; = < N >_u = \int_0^u [(\sigma_P(s,T) + \rho\sigma)^2 + \sigma^2(1-\rho^2)]ds. \tag{26}$$

We prove below how to obtain the following closed form of $\mathcal{E}$ using Girsanov's theorem and time change techniques:

$$\mathcal{E} \;\; = \;\; P(0,T)L_T^g \,\mathrm{N}\left(\frac{\ln\left(\frac{\mathrm{P(0,T)L_T^g}}{\mathrm{A_0}}\right) + \frac{1}{2}\xi(\mathrm{t_i})}{\sqrt{\xi(\mathrm{t_i})}}\right) \; - \; \mathrm{A_0 N}\left(\frac{\ln\left(\frac{\mathrm{P(0,T)L_T^g}}{\mathrm{A_0}}\right) - \frac{1}{2}\xi(\mathrm{t_i})}{\sqrt{\xi(\mathrm{t_i})}}\right) \tag{27}$$

Let us first write $\mathcal{E}$ as:

$$\mathcal{E} = P(0,T)\left(L_T^g \mathrm{Q}_T\left(\frac{A_{t_i}}{P(t_i,T)} < L_T^g\right) - \mathbb{E}_{\mathrm{Q}_T}\left[\frac{A_{t_i}}{P(t_i,T)}\mathbb{1}_{\left\{\frac{A_{t_i}}{P(t_i,T)} < L_T^g\right\}}\right]\right)$$

The key to the computation is the Dubins-Schwarz theorem (time change technique) which states that there exists a unique Q_T-Brownian motion B such that:

$$\forall u \in [0,T]\;,\; N_u = N_0 + B_{\xi(u)}. \tag{28}$$

Using this representation theorem, we get a new expression of the two parts of the expression $\mathcal{E}$:

$$\mathrm{Q}_T\left(\frac{A_{t_i}}{P(t_i,T)} < L_T^g\right) = \mathrm{Q}_T\left(N_{t_i} - \frac{1}{2}\xi(t_i) < \ln\left(\frac{P(0,T)L_T^g}{A_0}\right)\right)$$

$$\mathbb{E}_{\mathrm{Q}_T}\left[\frac{A_{t_i}}{P(t_i,T)}\mathbb{1}_{\left\{\frac{A_{t_i}}{P(t_i,T)} < L_T^g\right\}}\right] = \frac{A_0}{P(0,T)}\mathbb{E}_{\mathrm{Q}_T}\left[e^{N_{t_i} - \frac{1}{2}\xi(t_i)}\mathbb{1}_{\left\{N_{t_i} - \frac{1}{2}\xi(t_i) < \ln\left(\frac{P(0,T)L_T^g}{A_0}\right)\right\}}\right]$$

Since $N_{t_i} - \frac{1}{2}\xi(t_i) = B_{\xi(t_i)} - \frac{1}{2}\xi(t_i)$ is a Gaussian variable, it is then straightforward to obtain the formula (27) for $\mathcal{E}$. $\square$